On a small cancellation theorem of Gromov

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Abstract

We give a combinatorial proof of a theorem of Gromov, which extends the scope of small cancellation theory to group presentations arising from labelled graphs.

In this paper we present a combinatorial proof of a small cancellation theorem stated by M. Gromov in [Gro2], which strongly generalizes the usual tool of small cancellation. Our aim is to complete the six-line-long proof given in [Gro2] (which invokes geometric arguments).

Small cancellation theory is an easy-to-apply tool of combinatorial group theory (see [Sch] for an old but nicely written introduction, or [GH] and [LS]). In one of its forms, it basically asserts that if we face a group presentation in which no two relators share a common subword of length greater than 1/6 of their length, then the group so defined is hyperbolic (in the sense of [Gro1], see also [GH] or [Sh] for basic properties), and infinite except for some trivial cases.

The theorem extends these conclusions to much more general situations. Suppose that we are given a finite graph whose edges are labelled by generators of the free group F_m and their inverses (in a reduced way, see technical definition below). If no word of length greater than 1/6 times the length of the smallest loop of the graph appears on the graph, then the presentation obtained by taking as relations all the words read on all loops of the graph defines an hyperbolic group which (if the rank of the graph is at least $m+1$, to avoid trivial cases) is infinite. Moreover, the given graph naturally embeds isometrically into the Cayley graph.

The new theorem reduces to the classical one when the graph is a bouquet of circles. Noticeably, this criterion is as easy to use as the standard one.

For example, ordinary small cancellation theory cannot deal with such simple group presentations as $\langle\, S \mid w_1 = w_2 = w_3 \,\rangle$ because the two relators involved here, $w_1 w_2^{-1}$ and $w_1 w_3^{-1}$, share a long common subword. The new theorem can handle such situations: for "arbitrary enough" words w_1, w_2, w_3, such presentations will define infinite, hyperbolic groups, although from the classical point of view these presentations satisfy a priori only the $C'(1/2)$ condition from which nothing could be deduced.

Most importantly, this technique allows to (quasi-)embed prescribed graphs into the Cayley graphs of hyperbolic groups. It is the basic construction

involved in the announcement of a counter-example to the Baum-Connes conjecture with coefficients (see [HLS] which elaborates on [Gro2], or [Gh] for a survey). Indeed, this counter-example is obtained by constructing a finitely generated group (which is a limit of hyperbolic groups) whose Cayley graph quasi-isometrically contains a family of expanders.

1 Statement and discussion

Let S be a finite set which is the disjoint union of two sets S' and S'', with a bijection from S' to S'' called *being inverse*. The elements of S are called *letters*.

A *word* is a finite sequence of letters. The inverse of a word is the word made of the inverse letters put in reverse order.

A word is called *reduced* if it does not contain a letter immediately followed by its inverse.

A *labelled complex* is a finite unoriented 2-complex in which each oriented edge bears a letter, such that opposite edges bear inverse letters. (Each unoriented edge is considered as a couple of two oriented edges.) Thus each face defines a word (up to inversion and cyclic permutation) read on its boundary. We require a map of labelled complexes to preserve labels (but it may change orientation of faces, sending a face to a face with inverse boundary label — this amounts to considering maps between the corresponding oriented complexes).

A *labelled graph* is a 1-dimensional labelled complex.

A labelled complex is said to be *reduced* if there is no pair of oriented edges arising from the same vertex and bearing the same letter.

Note that a word can be seen as a (linear) labelled graph, which we will implicitly do from now on. The word is reduced if and only if the labelled graph is.

A *doublet* of a labelled complex is a word which has two different immersions in the labelled complex. (An immersion is a locally injective map of labelled complexes. Two immersions are considered different if they are different as maps of 2-complexes.) This is analogue to the traditionnal "piece" of small cancellation theory.

A *standard family of cycles* for a graph is a set of paths in the graph, generating the fundamental group, such that there exists a maximal subtree of the graph such that, when the subtree is contracted to a point (so that the graph becomes a bouquet of circles), the set of generating cycles is exactly the set of these circles. There always exists some.

We are now in a position to state the theorem.

Theorem 1 (M. Gromov) – *Let Γ be a reduced labelled graph. Let R be the set of words read on all cycles of Γ (or on a generating family of cycles). Let g be the girth of Γ and Λ be the length of the longest doublet of Γ.*

If $\Lambda < g/6$ then the presentation $\langle S \mid R \rangle$ defines a group G enjoying the following properties.

1. *It is hyperbolic, torsion-free.*

2. *If the rank of the fundamental group of Γ is greater than the number of generators, G is infinite and not quasi-isometric to $\mathbb{Z}$.*

3. *Any presentation of G by the words read on a standard family of cycles of Γ is aspherical (hence the cohomological dimension of G is at most 2).*

4. *The shortest relation in G is of length g.*

5. *For any reduced word w equal to e in G, some cyclic permutation of w contains a subword of a word read on a cycle of Γ, of length at least $(1 - 3\Lambda/g) > \frac{1}{2}$ times the length of this cycle.*

6. *The natural map from the labelled graph Γ into the Cayley graph of G is an isometric embedding.*

If Γ is a disjoint union of circles, this theorem almost reduces to ordinary $1/6$ small cancellation theory. The "almost" accounts for the fact that the length of a shared doublet between two relators is supposed to be less than $1/6$ the length of the smallest of the two relators in ordinary small cancellation theory, and less than $1/6$ the length of the smallest of all relators in our case. But it is clear from the proof below that the assumption in the theorem can be replaced by the following slightly weaker one: for each doublet, its length is less than $1/6$ the minimal length of the cycles of the graph on which the doublet appears. With this assumption, the theorem reduces to ordinary small cancellation when the graph is a disjoint union of circles.

The group obtained is not always non-elementary: for example, if there are three generators a, b, c and the graph consists in two points joined by three edges bearing a, b and c respectively, one obtains the presentation $\langle\, a, b, c \mid a = b = c \,\rangle$ which defines $\mathbb{Z}$. However, since the cohomological dimension is at most 2, it is easy to check (computing the Euler characteristic) that if the rank of the fundamental group of Γ is greater than the number of generators, then G is non-elementary.

This theorem is not stated explicitly in [Gro2] in the form we give but using a much more abstract formalism of "rotation families of groups". In the vocabulary thereof, the case presented here is when this rotation family contains only one subgroup of the free group (and its conjugates), namely the one generated by the words read on cycles of the graph with some base point; the corresponding "invariant line" U is the universal cover of the labelled graph Γ (viewed embedded in the Cayley graph of the free group). Reducedness of the labelling ensures quasi-convexity.

In [Gro2], this theorem is applied to a random labelling (or rather a variant of this theorem given below, in which reducedness is replaced with quasi-geodesicity). It is not difficult, using for example the techniques described in [Oll], to check that a random labelling satifies the small cancellation and quasi-geodesicity assumptions.

2 Proof

We now give some more definitions which are useful for the proof.

A *tile* is a planar labelled complex with only one face and no other edge than the boundary edges of this face (but not necessarily simply connected). By our definition of maps between labelled complexes, a tile is considered equal to the tile bearing the inverse word.

Convention: A tile may bear a word which is not simple (i.e. is a power of a smaller word). In this case the tile has a non-trivial automorphism. Say that on each boundary component of a tile we mark a starting point and that a map between tiles has to preserve marked points. This is useful for the study of torsion because with this convention asphericity of a presentation implies torsion-freeness and asphericity of the Cayley complex. Note that our definition of asphericity is thus slightly stronger than the one in [LS].

The *length* of a tile is the length of its boundary.

A *tile of a labelled complex* is any of its faces.

A *doublet* with respect to a set of tiles is a word which has immersions in the boundary two different tiles, or two distinct immersions in the boundary of one tile.

A *puzzle* with respect to a set of tiles is a planar labelled complex all tiles of which belong to this set of tiles (the same tile may appear several times in a puzzle).

A puzzle is said to be *minimal* if it has the minimal number of tiles for a given boundary word.

A puzzle is said to be *van Kampen-reduced* if there is no pair of adjacent faces such that the words read on the boundary of these two faces are inverse and the position (with respect to the marked point) of the letter read at a common edge of these faces is the same in the two copies of the boundary word of these faces (in other words, there is no trivial gluing). This corresponds to reduced van Kampen diagrams (see [LS]). (Incidentally, a reduced puzzle is van Kampen-reduced, though the converse is not necessarily true.)

Proof of the theorem – Let Γ be a reduced labelled graph. Γ defines a presentation R by taking all the words read along its cycles. The group presented by $\langle S \mid R \rangle$ will be the same if we take not all cycles but only a generating set of cycles.

Note that the assumption on doublets implies that no two distinct cycles of Γ bear the same word. We will implicitly use this fact below for unicity of lifts to Γ.

The fundamental group of the graph Γ is a free group. Let $\mathcal{C}$ be a finite generating set of $\pi_1(\Gamma)$ (maybe not standard). Let R be the set of words

read on the cycles in $\mathcal{C}$.

Add 2-faces to Γ in the following way: for each cycle in $\mathcal{C}$, glue a disk bordering this cycle.

Note Γ again this 2-complex. As the cycles in $\mathcal{C}$ generate all cycles, Γ is simply connected. If $\mathcal{C}$ is taken standard, Γ has no homotopy in degree 2.

Let D be a simply connected puzzle with respect to the tiles of Γ. We are going to show that there exists a constant $C > 0$ such that for any D, if D is van Kampen-reduced, then D satisfies a linear isoperimetric inequality $|\partial D| \geqslant C\,|D|$ where $|\partial D|$ is the boundary length of D and $|D|$ is the number of faces of D. This implies hyperbolicity (see for example [Sh]).

We can safely assume that all edges of D lie on some face (there are no "filaments"). Indeed, filaments only improve isoperimetry. Generally speaking, in what follows we will never mention the possible occurrence of filaments, their treatment being immediate. In particular, we suppose that all vertices of Γ are adjacent to at least two edges (or, equivalently, that any edge is adjacent to some face).

Let e be an internal edge of D, between faces f_1 and f_2. As D is a puzzle over the tiles of Γ, there are faces f'_1 and f'_2 of Γ bearing the same boundary word as f_1 and f_2 respectively (maybe up to inversion). (Note that these faces are unique, since no two faces of Γ can bear the same word as it would contradict the $1/6$ condition on doublets.)

The edge e belongs to f_1 and f_2 and thus can be lifted in Γ either in f'_1 or in f'_2. Say e is an *edge originating from* Γ if these two lifts coincide, so that in Γ, the two faces at play are adjacent along the same edge as they are in D.

Any labelled complex with respect to the tiles of Γ, all edges of which originate from Γ, can thus be lifted to Γ by lifting each of its edges.

Note that D is van Kampen-reduced if and only if there is no edge e originating from Γ and adjacent to faces f_1, f_2 such that $f'_1 = f'_2$.

We work by first proving the isoperimetric inequality for puzzles having all edges originating from Γ. Second, we will decompose the graph D into parts having all their edges originating from Γ and show that these parts are in $1/6$ small cancellation with each other. Then we will use ordinary small cancellation theory to conclude.

We begin by proving what we want for some particular choice of R.

Lemma 2 – *Let $\Delta = \operatorname{diam}(\Gamma)$. Suppose that R was chosen to be the set of words read on closed paths embedded in Γ of length at most 3Δ. Then, for any closed path in Γ labelling a reduced word w, there exists a simply connected puzzle with boundary word w, with tiles having their boundary words in R, all edges of which originate from Γ, and with at most $3\,|w|\,/g$ tiles.*

Proof of Lemma 2 – If $w \leqslant 2\Delta$ then by definition of R there exists a one-tile puzzle spanning w, and as $|w| \geqslant g$ the conclusion holds. Show by

induction on n that if $|w| \leqslant n\Delta$ there exists a puzzle D spanning w with at most n pieces. This is true for $n = 2$. Suppose this is true up to $n\Delta$ and suppose that $2\Delta \leqslant |w| \leqslant (n+1)\Delta$.

Let $w = w'w''$ where $|w'| = 2\Delta$. As the diameter of Γ is Δ, there exists a path in Γ labelling a word x joining the endpoints of w', with $|x| \leqslant \Delta$. So $w'x^{-1}$ is read on a cycle of Γ of length at most 3Δ, hence (its reduction) belongs to R. Now xw'' is a word read on a cycle of Γ, of length at most $|w| - \Delta \leqslant n\Delta$. So there is a puzzle with at most n tiles spanning xw''. Gluing this puzzle with the tile spanning $w'x^{-1}$ along the x-sides provides the desired puzzle. (Note that this gluing occurs in Γ, so that edges of the resulting puzzle originate from Γ.)

So for any w we can find a puzzle spanning it with at most $1 + |w|/\Delta$ tiles. As $\Delta \geqslant g/2$ and as $|w| \geqslant g$, we have $1 + |w|/\Delta \leqslant 1 + 2|w|/g \leqslant 3|w|/g$. □

Corollary 3 – *For any choice of R, there exists a constant α such that any minimal simply connected puzzle D with respect to the tiles of Γ all internal edges of which originate from Γ satisfies the isoperimetric inequality $|\partial D| \geqslant \alpha |D|$.*

Proof of Corollary 3 – Indeed, the existence of an isoperimetric constant does not depend on the presentation. □

The second lemma is just ordinary small cancellation theory (see for example the appendix of [GH], or [LS]), stated in the form we need. We include a short proof here for completeness.

Lemma 4 – *Let R be a set of simply connected reduced tiles. Suppose that any doublet with respect to two tiles $t, t' \in R$ is a word of length at most λ times the smallest boundary length of t and t', for some constant $\lambda < 1/6$.*

Then any simply connected van Kampen-reduced puzzle D with respect to the tiles of R satisfies the following properties.

1. *The reduction w of the boundary word of D contains a subword of some tile t with length at least $(1 - 3\lambda) > \frac{1}{2}$ times the boundary length of t.*

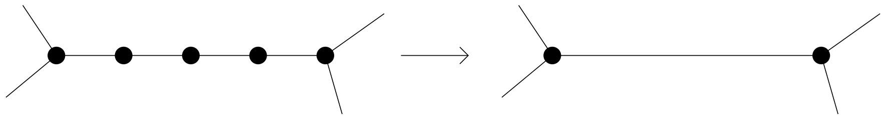
2. *The word w is not a proper subword of the boundary word of some tile.*

3. *The boundary length $|\partial D|$ is at least $1 - 6\lambda$ times the sum of the lengths of the faces of D, and at least the boundary length of the largest tile it contains.*

The value $1/6$ is optimal, as a hexagonal tiling shows.

Proof – Let D be a van Kampen-reduced puzzle w.r.t. R. We can suppose that its boundary word is reduced (isoperimetry for the reduction implies isoperimetry for D a fortiori). It is a planar graph. Define a metric graph D' as follows: start with D, but for each pair of adjacent faces, replace all consecutive edges between these two faces by a single edge. Define the length of this new edge as the number of edges it replaces. So D' is a metric planar

graph having the same faces as D but in which every vertex is of degree at least 3. The small cancellation assumption states that the length of any edge in D' is at most λ times the smallest boundary length of the two adjacent faces.



Let V, E and F denote the number of vertices, edges and faces of D', respectively. Let E_i and E_e denote the number of internal and external edges, and F_i, F_e the number of internal faces and faces adjacent to the boundary. Let F_e^1 and F_e^2 denote the number of external faces with exactly one, or at least two, external edges, respectively.

As every vertex has degree at least 3, we have $E \geqslant 3V/2$. By definition we have $E_e \geqslant F_e^1 + 2F_e^2$. Since $\lambda < 1/6$, any internal face has at least 6 edges.

We now prove that there are at least two external faces with exactly one external edge and at most three internal edges (this will prove the first assertion of the lemma). Let F_e' and F_e'' be the number of external faces with exactly one external edge and, respectively, at most three or at least four internal edges. Since any internal face has at least six internal edges, and since any external face with two external edges has at least two internal edges, we get $E_i \geqslant \frac{1}{2}\left(F_e' + 4F_e'' + 2F_e^2 + 6F_i\right)$.

The Euler formula writes

$$\begin{aligned}
1 &= V - E + F = V - \frac{2}{3}E - \frac{1}{3}E + F \\
&\leqslant 0 - \frac{1}{3}E_i - \frac{1}{3}E_e + F_e + F_i \\
&\leqslant -\frac{1}{6}\left(F_e' + 4F_e'' + 2F_e^2 + 6F_i\right) - \frac{1}{3}\left(F_e' + F_e'' + 2F_e^2\right) + F_e' + F_e'' + F_e^2 + F_i \\
&= \frac{1}{2}F_e'
\end{aligned}$$

Hence there exist at least two faces with exactly one external edge and at most three internal ones.

By the small cancellation assumption, the internal edges of such a face have cumulated length at most 3λ times the length of this face. This proves the first assertion of the lemma.

The second assertion is easy. Suppose that the boundary word of some puzzle is a proper subword w of the boundary word x of some tile t. We know that w contains a subword which contains a proportion at least $(1-3\lambda) > 1/2$ of the boundary word of some tile t'. If $|w| \leqslant |x|/2$ we have $t' \neq t$, so t and t' share a common subword of length more than one half the boundary length of t', which contradicts the small cancellation assumption. If $|w| > |x|/2$, then define a new puzzle by gluing the inverse tile t^{-1} to the original puzzle (and

reducing the resulting puzzle): the new puzzles now contains the reduction of the word wx^{-1}, which is at least one half of the boundary word of t^{-1}; use the same argument.

Isoperimetry follows immediately. Let F be a external face of D' with at most three internal edges. Let ℓ be the length of F. Let D'' be the (maybe non connected) graph obtained from D' by removing F. We have $|\partial D'| = |\partial D''| + \ell - 2\,|F \cap D''| \geqslant |\partial D''| + (1-6\lambda)\ell$ hence the conclusion by backwards induction on the number of faces.

The fact that the boundary length of D is at least the boundary length of the largest face is trivial if D has only one face, and otherwise follows from the fact that there are two external faces having at least a proportion $1-3\lambda > \frac{1}{2}$ of their length on the boundary. Remove such a face (but not the largest one): this decreases the boundary length. Iterate this process up to a diagram containing only the largest face. □

Corollary 5 – *Let R be a set of (not necessarily simply connected) reduced tiles. Suppose that any doublet with respect to two tiles $t, t' \in R$ is a word of length at most λ times the smallest length of the boundary component of t and t' it immerses in, for some constant $\lambda < 1/6$.*

Then, any simply connected puzzle with respect to this set of tiles contains only simply connected tiles.

Proof of the corollary – Let D be a simply connected puzzle with respect to R. Let t be a non-simply connected tile in D. Consider the subdiagram D' of D filling one of the holes of t. We can suppose that D' contains only simply connected tiles (otherwise, take again a non-simply connected tile in D' and a diagram filling one of its holes, and repeat this operation as many times as necessary).

By the previous lemma, the boundary word w of D' contains a subword w' of the boundary word $w_{t'}$ of some tile t' of D', with $|w'| \geqslant |w_{t'}|/2$. But as all tiles are reduced, the boundary word of D' is the word read on the boundary component of t spanning D'. So t' and t have in common a word of length at least one half the boundary length of t', which contradicts the assumption. □

Back to our simply connected puzzle D with tiles in Γ. A puzzle is built by taking the disjoint union of all its tiles and gluing them along the internal edges.

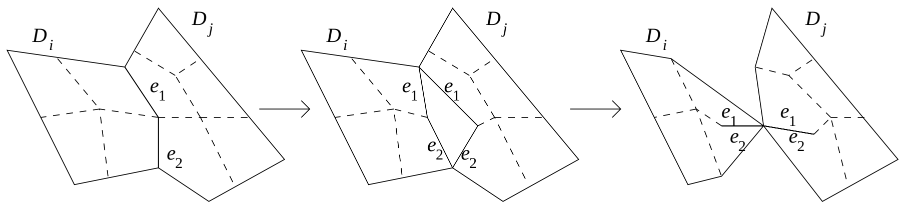
First, define a (maybe not connected) puzzle D' by taking the disjoint union of all tiles of D and gluing them along the internal edges of D originating from Γ. All internal edges of D' originate from Γ.

As D is van Kampen-reduced, D' is as well.

Let D_i, $i = 1, \ldots, n$ be the connected components of D'. They form a partition of D. The puzzle D is obtained by gluing these components along the internal edges of D not originating from Γ.

It may be the case that the boundary word of some D_i is not reduced. This means that there is a vertex on the boundary of D_i which is the origin of two (oriented) edges bearing the same vertex. We will modify D in order to avoid this. Suppose some D_i has non-reduced boundary and consider two edges e_1, e_2 of D responsible for this: e_1 and e_2 are two consecutive edges with inverse labels. These edges are either boundary edges of D or internal edges. In the latter case this means that D_i is to be glued to some D_j. We treat only this latter case as the other one is even simpler.

Make the following transformation of D: do not glue any more edge e_1 of D_i with edge e_1 of D_j, neither edge e_2 of D_i with edge e_2 of D_j, but rather glue edges e_1 and e_2 of D_i, as well as edges e_1 and e_2 of D_j, as in the following picture. This is possible since by definition e_1 and e_2 bear inverse labels.



This kind of operation has been studied and termed *diamond move* in [CH].

Since Γ is reduced, the lifts to Γ of the edges e_1 and e_2 of D_i are the same edge of Γ. This shows that the transformation above preserves the fact that all edges of D_i and of D_j originate from Γ.

The resulting diagram (denoted D again) has the same number of faces as before, and no more boundary edges. Thus, proving isoperimetry for the modified diagram will imply isoperimetry for the original one as well. So we can safely assume that the boundary words of the D_i's are reduced.

Now consider D as a puzzle with the D_i's as tiles. Note that these tiles are not necessarily simply connected.

These tiles satisfy the condition of Corollary 5. Indeed, suppose that the two tiles D_i, D_j are to be glued along a common (reduced!) word w. By definition of the D_i's, the edges making up w do not originate from Γ.

As the edges of D_i originate from Γ, there is a lift $\varphi_i : D_i \to \Gamma$ (as noted above). Consider the two lifts $\varphi_i(w)$ and $\varphi_j(w)$. As the edges making up w do not originate from Γ, these two lifts are different. As w is reduced these lifts are immersions. So w is a doublet. By assumption the length of w is at most $\Lambda < g/6$.

Now as D_i lifts to Γ, any boundary component of D_i goes to a closed path in the 1-skeleton of Γ. This proves that the length of any boundary component of D_i is at least g.

So the tiles D_i satisfy the small cancellation condition with $\lambda = \Lambda/g < 1/6$. As they are tiles of a simply connected puzzle, by Corollary 5 they are simply connected.

Then by Lemma 4, the boundary of D is at least $1 - 6\lambda$ times the sum of the boundary lengths of the D_i's. Since D is minimal, each D_i is as well, and as D_i is simply connected, by Corollary 3 it satisfies the isoperimetric inequality $|\partial D_i| \geqslant \alpha\,|D_i|$. So

$$|\partial D| \geqslant (1 - 6\lambda) \sum |\partial D_i| \geqslant \alpha(1 - 6\lambda) \sum |D_i| = \alpha(1 - 6\lambda)\,|D|$$

which shows the isoperimetric inequality for D, hence hyperbolicity.

For asphericity and the cohomological dimension (hence torsion-freeness), suppose that R is standard (so that Γ is aspherical) and that there exists a van Kampen-reduced spherical diagram D. Define the D_i's as above. As we shown above that the boundary length of D (which is 0) is at least the boundary length of any D_i, this means that the boundary length of all the D_i's is 0. Hence each D_i is a spherical diagram itself. But by definition D_i lifts to Γ. We conclude with the following lemma.

Lemma 6 – *Suppose that the set of paths read along faces of Γ is standard. Let D be a non-empty spherical puzzle all edges of which come from Γ. Then D is not reduced.*

Proof of the lemma – Let T be a maximal tree of the 1-skeleton of Γ witnessing for standardness of the family of cycles. Homotope T to a point. This turns Γ into a bouquet of circles with a face in each circle. Similarly, homotope to a point any edge of D coming from a suppressed edge of Γ. This way we turn D into a spherical van Kampen diagram with respect to the presentation of the fundamental group of Γ (i.e. the trivial group) by $\langle\, c_1, \ldots, c_n \mid c_1 = e, \ldots, c_n = e \,\rangle$. But there is no reduced spherical van Kampen diagram with respect to this presentation, as can immediately be checked. □

The last affirmations of the theorem follow easily from the last affirmations of Lemma 4. The smallest relation in the group presented by $\langle\, S \mid R \,\rangle$ is the boundary length of the smallest puzzle, which by Lemma 4 is at least the smallest boundary length of the D_i's, which is at least g. Similarly, any reduced word representing the trivial element in the group is the boundary of a reduced diagram, thus contains as a subword at least one half of the boundary word of some D_i.

For the isoperimetric embedding of Γ in the Cayley graph of the group, suppose that some geodesic path in the graph labelling a word x is equal to a shorter word y in the quotient. This means that there exists a puzzle with boundary word xy^{-1}, made up of tiles with cycles of Γ as boundary words. Now (if Γ contains no filaments) x is part of some cycle labelled by $w = xz$ of the graph. Since the path x is of minimal length, we have $|x| \leqslant |z|$. So $|xy^{-1}| < |w|$ and in particular, the puzzle bordering xy^{-1} cannot contain a tile with boundary word w. Glue a tile with boundary word $(xz)^{-1}$ to the puzzle xy^{-1} along the subwords x and x^{-1}. This results in a (reduced)

puzzle bordering $z^{-1}y^{-1}$, containing a tile $z^{-1}x^{-1}$. This is impossible as by assumption $|y| < |x|$.

Non-elementariness in case the rank of the fundamental group of Γ is large enough follows immediately from a computation of the Euler characteristic, using that the cohomological dimension is at most 2.

This proves the theorem. $\square$

3 Further remarks

Remark 7 – The proof above gives an explicit isoperimetric constant when the set of relators taken is the set of all words read on cycles of the graph of length at most three times the diameter: in this case, any minimal simply connected puzzle satisfies the isoperimetric inequality

$$|\partial D| \geqslant g(1-6\Lambda/g)\,|D|\,/3$$

This explicit isoperimetric constant growing linearly with g (i.e. "homogeneous") can be very useful if one wants to apply such theorems as the local-global hyperbolic principle, which requires the isoperimetric constant to grow linearly with the sizes of the relators.

Remark 8 – The assumption that Γ is reduced can be relaxed a little bit, provided that some quasi-geodesicity assumption is granted, and that the definition of a doublet is emended.

Redefine a *doublet* to be a couple of words (w_1, w_2) such that both immerse in Γ and such that $w_1 = w_2$ in the free group. The *length* of a doublet (w_1, w_2) is the maximal length of w_1 and w_2.

There are trivial doublets, for example if $w_1 = w_2$ and both have the same immersion. However, forbidding this is not enough: for example, if a word of the form $aa^{-1}w$ immerses in the graph, then $(aa^{-1}w, w)$ will be a doublet.

A *trivial doublet* is a doublet (w_1, w_2) such that there exists a path p in Γ joining the beginning of the immersion of w_1 to the beginning of the immersion of w_2 such that p is labelled with a word equal to e in the free group.

The new theorem is as follows.

Theorem 9 (M. Gromov) – *Let Γ be a labelled graph. Let R be the set of words read on all cycles of Γ (or on a generating family of cycles). Let g be the girth of Γ and Λ be the length of the longest non-trivial doublet of Γ.*

Suppose that $\lambda = \Lambda/g$ is less than $1/6$.

Suppose that there exist a constant $A > 0$ such that any word w immersed in Γ of length at least L satisfies $\|w\| \geqslant A(|w| - L)$ for some $L < (1-6\lambda)g/2$.

Then the presentation $\langle\, S \mid R \,\rangle$ defines a hyperbolic, infinite, torsion-free group G, and (if R is standard) this presentation is aspherical (hence the

cohomological dimension of G is at most 2). Moreover, the natural map of labelled graphs from Γ to the Cayley graph of G is a $(1/A, AL)$-quasi-isometry. The shortest relation of G is of length at least $Ag/2$, and any reduced word equal to e in G contains as a subword the reduction of at least one half of a word read on a cycle of Γ.

(In the notation of [GH], by a (λ, c)-quasi-isometry we wean a map f such that $d(x,y)/\lambda - c \leqslant d(f(x), f(y)) \leqslant \lambda d(x,y) + c$.)

Remark 10 – The same theorems hold if we use the $C(7)$ condition instead of the $C'(1/6)$ condition, but in this case there is no control on the radius of injectivity.

Remark 11 – Using the techniques in [Del] or [Oll], the same kind of theorem holds starting with any torsion-free hyperbolic group instead of the free group, provided that the girth of the graph is large enough w.r.t. the hyperbolicity constant, and that the labelling is quasi-geodesic.

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